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X.

CONTRIBUTIONS FROM PHYSICAL LABORATORY OF HARVARD COLLEGE.

No. XII. — ON VORTEX RINGS IN LIQUIDS.

By John Trowbridge.

Presented, March 14, 1877.

It has often been observed by chemists that a drop of colored liquid falling from a burette into a liquid of a different specific gravity, in which it can diffuse, assumes the form of a ring. Vortex motion, by the researches of Helmholtz, Thomson, Rankine, and Maxwell, is now attracting so much attention, that I have thought that a study of the general equations of motion of matter in connection with a study of these rings would contribute to our knowledge of vortex movement.

Prof. W. B. Rogers published in "The American Journal of Arts and Sciences," Vol. XXVI., 1858, a paper on smoke rings and liquid rings, and described several methods of studying them. In Professor Tait's "Recent Advances in Physical Science," a method of forming smoke rings is given. The apparatus consists merely of a large box closed at one end by a thin sheet of india rubber, or with a tightly stretched towel, and having a circular opening of six or eight inches in diameter at the other. Clouds of sal-ammoniac vapor are generated inside the box, and rings are expelled from the circular opening by a blow upon the rubber or towel. Sir William Thomson suggests that two such boxes placed so that the rings may impinge on each other at any angle would form a useful apparatus for studying the behavior of such rings towards each other. At the conclusion of this paper, several methods of studying liquid rings will be described. When a drop of liquid falls from a short distance into a liquid of less density, in which it cannot diffuse, the conditions of its motion just after the instant of its striking the surface of the liquid of less density are indicated by the general equations of heterogeneous strains.* "For each particle we have the component velocities u, v, w, parallel to the fixed axes OX, OY, OZ. These have the following expressions: —

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* Thomson and Tait's Natural Philosophy.

$$\text{Eq. (1)}: \quad u = \frac{d\alpha}{dt}, \qquad v = \frac{d\beta}{dt}, \qquad w = \frac{d\gamma}{dt};$$

x, y, z, t being independent variables, and α, β, γ, functions of them. If the disturbed condition is so related to the initial condition that every portion of the body can pass from its initial to its disturbed position and strain, by a translation and a strain without rotation, — *i.e.*, if the three principal axes of the strain at any point are lines of the substance which retain their parallelism, — we must have, —

$$\text{Eq. (2)}: \quad \frac{d\beta}{dz} = \frac{d\gamma}{dy}, \qquad \frac{d\gamma}{dx} = \frac{d\alpha}{dz}, \qquad \frac{d\alpha}{dy} = \frac{d\beta}{dx};$$

and, if these equations are fulfilled, the strain is now rotational, as specified." But these equations express that $\alpha\, dx + \beta\, dy + \gamma\, dz$, is the differential of a function of three independent variables; and therefore, in order that there may be no rotation, a strain potential must exist. The forces which solicit the particles of the drop when it rests upon the liquid of less density in which it cannot diffuse are evidently their mutual attraction, a force arising from the superficial tension of the liquid, and one arising from gravitation. It is evident, from a consideration of these forces, that, after the drop has suffered a strain at the surface, every portion of the drop cannot pass from its initial position to the next following by a translation and a strain without rotation. For the drop tends to return from a shape approaching an oblate spheroid to that of a sphere. Equations (2) do not hold, and a strain potential does not exist, and therefore this drop must rotate. This rotation is not in general of the ring form. If, on the other hand, the drop of liquid can diffuse itself in the liquid through which it falls, each particle with the velocity u, v, w, is solicited at the moment of impact by a superficial tension, by the force of gravitation, and by a force arising from the rate of diffusion. In this case, there is no tendency of the body to reassume the spheroidal form in its passage through the liquid. On the other hand, to assume that each particle in the next state of the drop very near that which it assumes on striking the free surface of the liquid of less density, is translated without rotation, is to assume that each particle is compelled to move in restrained limits, which do not exist. For the components X^1, Y^1, Z^1, of the attraction, which tend to make the non-diffusible drop reassume its spherical form, we have in the case of the diffusible the components

X, Y, Z, of an external force arising from the superficial tension of the liquid, and the impulse given to the drop.

If we follow the notation of Poisson * and Helmholtz,† we shall have for the general equations of internal motion of a liquid : —

$$\left.\begin{aligned} X-\frac{1}{h}\frac{dp}{dx}&=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\\ Y-\frac{1}{h}\frac{dp}{dy}&=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\\ Z-\frac{1}{h}\frac{dp}{dz}&=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\end{aligned}\right\}3$$

$$\frac{dh}{dt}+u\frac{dh}{dx}+v\frac{dh}{dy}+w\frac{dh}{dz}=\frac{\delta h}{\delta t}\quad(4)$$

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=o\quad(5)$$

In which p is the pressure in a liquid at the point x, y, z; X, Y, Z, are the components of the external forces acting on a unit of mass; and h is the density. When the variation of h is infinitely small, we have Eq. (5). The forces X, Y, Z, are considered to have a potential V. So that

$$X=\frac{dV}{dx},\qquad Y=\frac{dV}{dy},\qquad Z=\frac{dV}{dz}\quad\text{Eq. (6)}$$

and the velocities u, v, w, a velocity potential φ. So that

$$u=\frac{d\phi}{dx},\qquad v=\frac{d\phi}{dy},\qquad w=\frac{d\phi}{dz}\quad\text{Eq. (7)}$$

or,
$$udx+vdy+wdz=d\varphi,$$

and φ satisfying the equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=o$$

which is what equation (5) becomes under the conditions expressed above. We must therefore have

$$\frac{du}{dy}=\frac{dv}{dx},\qquad \frac{dv}{dz}=\frac{dw}{dy},\qquad \frac{dw}{dx}=\frac{du}{dz}\quad\text{Eq. (8)}$$

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* Traité de Mechanique.

† Crelle's Journ., LV., 1858.

equations similar to the equations expressing a strain potential. Helmholtz has shown that in the case of rotation of a fluid element, Eqs. (8) become

$$\left.\begin{array}{l} \frac{dv}{dz} - \frac{dw}{dy} = 2\xi \\ \frac{dw}{dx} - \frac{du}{dz} = 2\gamma \\ \frac{du}{dy} - \frac{dv}{dx} = 2\zeta \end{array}\right\} \text{Eq. 9}$$

and therefore "the existence of a velocity potential is inconsistent with the existence of rotation of the fluid element." We have seen from the equations of strain that the existence of a strain potential is inconsistent with the rotation of a material particle. Let us now see if vortex movement can arise in a liquid from variation of density and pressure. Following Helmholtz's notation, we have, if ψ is a function of x, y, z, t,

$$\frac{\delta\psi}{\delta t} = \frac{d\psi}{dt} + u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz} \quad \text{Eq. (10)}$$

Calling ξ, γ, ζ, the components of the angular velocity, we can obtain their variations by substituting them in succession in Eq. (10). If we eliminate X, Y, Z, from Eqs. (3) by the help of Eqs. (6), supposing that h and p are functions of x, y, z, t, we obtain, introducing the values of ξ, γ, ζ, from Eqs. (9): —

$$\text{Eq. (11)} \quad \frac{\delta\xi}{\delta t} = -\xi\left(\frac{dv}{dy} + \frac{dw}{dz}\right) + \gamma\frac{dv}{dx} + \zeta\frac{dw}{dx} + \frac{1}{2h^2}\left(\frac{dh}{dz}\cdot\frac{dp}{dy} - \frac{dh}{dy}\,\frac{dp}{dz}\right)$$

and similar expressions for the variations of γ and ζ. If the variation of h is infinitely small, we obtain by the aid of Eq. (5): —

$$\frac{\delta\xi}{\delta t} = \xi\frac{du}{dx} + \gamma\frac{dv}{dx} + \zeta\frac{dw}{dx}$$

If it is not infinitely small, we have the term

$$\frac{1}{2h^2}\left(\frac{dh}{dz}\,\frac{dp}{dy} - \frac{dh}{dy}\,\frac{dp}{dz}\right)$$

which is independent of ξ, γ, ζ, and depends upon the variation of h and p. This term enters into the expressions for the variations in the angular velocities; and shows, therefore, that a vortex movement can arise in a process of diffusion by a variation in density and pressure, without the aid of initial angular velocities. This condition can be shown experimentally by dropping a somewhat dense solution of one of the aniline colors into a mixture of glycerine and water. The original ring, after ceasing to move downward in the mixture, breaks up gradually into segments, which slowly in their turn assume the ring form. A mixture of water and glycerine is not necessary: peculiar cusp-like figures indicating the first stage of vortical movement can be seen whenever a thin stratum of one liquid slowly diffuses itself through another liquid of different density.

By a consideration of the equations —

$$\varepsilon\,\xi + (u_1 - u)dt = \varepsilon\left(\xi + \frac{\delta\xi}{\delta t}\,dt\right)$$

$$\varepsilon\,\gamma + (v_1 - v)dt = \varepsilon\left(\gamma + \frac{\delta\gamma}{\delta t}\,dt\right)$$

$$\varepsilon\,\zeta + (w_1 - w)dt = \varepsilon\left(\zeta + \frac{\delta\zeta}{\delta t}\,dt\right)$$

given by Helmholtz, from which he draws the conclusion that "each vortex line remains continually composed of the same elements of fluid, and swims forward with them in the fluid," we see, on introducing the new expressions which we have found for $\frac{\delta\xi}{\delta t}$, &c., Eq. (11), that we approach nearer and nearer to this theoretical conclusion when the variations of h are smaller and smaller. Obviously, we should then obtain the most perfect rings when the drop and the liquid in which the motion takes place are composed of the same liquid. And, *therefore, a drop of water falling into water must form a more perfect ring than that formed by a drop of any colored liquid of greater density than water.*

The formation of these liquid rings is as fascinating and as simple an occupation as blowing soap-bubbles. All liquids falling from such a height that the surface of the liquid is not too much disturbed to enable the drop to be acted upon symmetrically by the forces at the free surface will form rings, if too great differences of density do not exist, and if the drop can diffuse in the liquid. The preceding mathematical discussion, as we have seen, shows us that a drop of pure water on striking the same element under the above conditions must necessarily assume the ring shape. This can be shown experimentally by covering the free surface of the water with a fine powder, or with matter in a fine state of subdivision. I have found that an alcoholic tincture of ginger, which gives on the surface of water a milky liquid consisting of particles in a fine state of subdivision, answers the purpose very well. Fine particles will be carried down by the drop, and will be seen to rotate in a vortex ring far below the surface. This fact can be stated, also, by the employment of any of the aniline colors which are solvent in water, the falling drop consisting of a colored solution whose specific gravity does not differ sensibly from that of water. The method that I have employed to produce the rings consists merely of a small glass tube, slightly smaller at one end than the other. A bit of cotton is wedged in nearer the larger end, over which a piece of flexible rubber tubing is slipped. With the aid of the mouth, one can fill this tube with liquid and eject it in drops at pleasure. The same apparatus enables us to form the rings beneath the surface of the liquid. With a tube bent horizontally, one can send the rings through a liquid in any desired direction; and, by means of a three-way glass joint and a small india-rubber bag, one can send forth, by the same impulse, two rings whose paths make any desired angle with each other. By partly immersing the glass tubes connected with the three-way tube in the free surface of the liquid, and covering the surface of the water with fine powder, one can study the mutual behavior of half-vortex rings. A simpler method is to illuminate, by means of a gas-light, the bottom of a flat, white porcelain dish filled with water, and to observe the shadows of the half-vortex rings on the bottom of the dish formed by the movement of two spatulæ along the surface. It can be readily seen, by this simple method, that a half-vortex ring moving near another in a parallel path and with a less velocity tends to follow in the path of the first; and that two equal half-vortex rings moving in opposite directions along the same path separate into two vortices which move at right angles to the path of the original vortices. We can conclude, also, from this general discussion, that, whenever a mass of vapor of greater density than the surrounding air is suddenly formed in the higher regions of the atmosphere, it tends to descend through it in a vortex ring.

The results of the preceding discussion are as follows: —

1. An analogy between the strain potential and the velocity potential is indicated.

2. It is shown that the formation of liquid rings is a necessary result of the fundamental equations of strains and those of hydrodynamics; and that they constitute a general and not a special phenomenon. A drop of water falling into water from a suitable height must assume a ring shape.

3. Vortices can and do arise in certain processes of diffusion.

4. Simple methods of studying vortex motion in liquids are given.